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UPPER AND LOWER BOUNDS TO EIGENVALUES BY WEIGHTING FUNCTION APPROXIMATIONS

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Abstract—A method is developed making use of variational principles and Rayleigh's quotient which yields lower bounds to eigenvalues. The method is the counterpart of the Rayleigh–Ritz method in the sense that the results obtained from both methods will improve, i.e. approach to the exact value, as more and more terms are considered, both rely on variational principles, they are similar systematically and conceptually, and this method yields lower bounds to eigenvalues which cannot be obtained from the Rayleigh–Ritz method. Therefore, with the results from both methods the eigenvalues can be bracketed into a small region. The most important advantage of the method is the lower bounds to all eigenvalues can be obtained from the solution of one transcendental equation.

NOTATION

A	cross-sectional area
A_i	unknown coefficients in solution of differential equations
a, b	end points of the region
C_i	minimum value of inner product
c_i^p, d_i^p	constant values of piecewise continuous step functions
e_i	subdivision points of the region
E	Young's modulus
F	function in separation of variables technique
f, f_i	functions in differential equation
g, g_i	functions in differential equation
g_1, g_2	solution of differential equation in Graetz problem
h, h_i	functions in differential equation
I	area moment of inertia
i, r	indices
J_0, J_1, Y_0, Y_1	Bessel functions
k_i^4	constant value of step function in bending problem
K, K', K^*, M, M'	differential and/or algebraic operators
L	length of beam and column
n	number of terms and subdivisions
p	order of differential equation
P	buckling load on column
$R(\phi)$	Rayleigh's quotient
s	thickness of column and beam
s_0	initial thickness of column and beam
x	independent variable
u	function multiplying operator in differential equation
y, v	functions
y_1, y_2	solution of differential equation in bending problem
α, α_1	taper coefficient of variable thickness column and beam
β, γ	piecewise continuous step functions
λ^2, λ_i^2	eigenvalues of Graetz and buckling problems and general second order differential equation
$\bar{\lambda}^2, \bar{\lambda}_1^2$	eigenvalues of the system subject to constraints
$\lambda_*^4, \lambda^4, \lambda_i^4$	eigenvalue of bending problem
μ^2, μ_i^2	eigenvalues of second order differential equation
υ^2, υ^4	upper bounds of eigenvalues
ρ	independent variable, mass density
η	independent variable
ϕ	admissible comparison function in Rayleigh's quotient and Rayleigh–Ritz method
ω	function multiplying operator in differential equation
Γ	function in separation of variables technique
τ	time
$\Omega^2, \Omega_i^2, \Omega^4, \Omega_i^4$	lower bounds to eigenvalues

INTRODUCTION

Consider an eigenvalue problem given by

$$K(f) - \lambda^2 M(f) = 0 \tag{1}$$

where

$$f = f(x) \quad a \leq x \leq b \tag{2}$$

with homogenous boundary conditions, where K is a differential operator and M is either a differential or algebraic operator of lower order than K. Let us assume that both K and M are positive definite and self-adjoint operators, and eqn (1) has an exact solution, therefore, the eigenvalues and eigenfunctions of this problem can be solved for directly. Let the equivalent variational formulation of this problem be[1]

$$\lambda_i^2 = \min \frac{\langle f_i, Kf_i \rangle}{\langle f_i, Mf_i \rangle} \quad \text{for } i = 1, 2, 3, \ldots, \infty \tag{3}$$

where f_i is orthogonal to f_j with respect to M. This orthogonality condition can be expressed as

$$\langle f_i, Mf_j \rangle = 0 \quad \text{for } j = 1, 2, \ldots, i-1. \tag{4}$$

The notation $\langle v, y \rangle$ designates the integration over x in the given region.

$$\langle v, y \rangle = \int_a^b vy \, \mathrm{d}x \tag{5}$$

where

$$\begin{aligned} v &= v(x) \quad a \leq x \leq b \\ y &= y(x) \quad a \leq x \leq b. \end{aligned} \tag{6}$$

For the above problem the Rayleigh's quotient is given by

$$v^2 = R(\phi) = \frac{\langle \phi, K\phi \rangle}{\langle \phi, M\phi \rangle}. \tag{7}$$

Rayleigh's quotient is a functional that depends on the trial function ϕ, and has a stationary value in the neighborhood of an eigenfunction. The value obtained from Rayleigh's quotient is an upper bound to the eigenvalue[2]. If the trial function chosen happens to be an eigenfunction, Rayleigh's quotient will yield the corresponding eigenvalue.

At this point consider another eigenvalue problem

$$uK(g) - \mu^2 \omega M(g) = 0 \tag{8}$$

where

$$\begin{aligned} g &= g(x) \quad a \leq x \leq b \\ u &= u(x) \quad a \leq x \leq b \\ \omega &= \omega(x) \quad a \leq x \leq b \end{aligned} \tag{9}$$

with the same set of boundary conditions as of eqn (1). The equivalent variational formulation of this problem is

$$\mu_i^2 = \min \frac{\langle ug_i, Kg_i \rangle}{\langle \omega g_i, Mg_i \rangle} \quad \text{for } i = 1, 2, 3, \ldots, \infty \tag{10}$$

and Rayleigh's quotient is given by

$$\upsilon = R(\phi) = \frac{\langle u\phi, K\phi\rangle}{\langle \omega\phi, M\phi\rangle}. \tag{11}$$

In this problem the function $u(x)$ and $\omega(x)$ are the so called weighting functions and the introduction of them into the differential equation will give rise to an eigenvalue problem which cannot be solved directly. Upper bounds to the eigenvalues and approximations to the eigenfunctions of this problem can be obtained by the application of the Rayleigh–Ritz method. There are certain lower bounds methods which were developed for specific eigenvalue problems. The better known of these are the ones developed by Weinstein[3], Kato[4], Bazley and Fox[5], and Pneuli[6]. Their most important drawback is they can be applied to a very limited number of problems due to their restrictuons, difficulty in application, and mostly they are developed for a specific type of problem.

Consider the maximum–minimum characterization of eigenvalues. Rayleigh's theorem for one constraint[7] states that the first eigenvalue of a system which is subject to one constraint lies between the first and second eigenvalues of the same system without being subject to the constraint, i.e.

$$\lambda_1^2 \le \bar{\lambda}_1^2 \le \lambda_2^2 \tag{12}$$

where $\bar{\lambda}_i^2$ are the eigenvalues of the system subject to the constraint. As

$$\bar{\lambda}_1^2 = \min \frac{\langle uf_1, Kf_1\rangle}{\langle \omega f_1, Mf_1\rangle} \tag{13}$$

subject to the constraint we can write

$$\lambda_2^2 = \max\left\{\min \frac{\langle uf_2, Kf_2\rangle}{\langle \omega f_2, Mf_2\rangle}\right\} \tag{14}$$

where f_2 is subject to the same constraint which is

$$\langle f_2, \phi\rangle = 0. \tag{15}$$

Making use of Rayleigh's theorem for any number of constraints[8] we can generalize eqn (14) as

$$\lambda_r^2 = \max\left\{\min \frac{\langle uf_r, Kf_r\rangle}{\langle \omega f_r, Mf_r\rangle}\right\} r = 2, 3, \ldots, \infty \tag{16}$$

subject to r-1 constraints.

LOWER BOUNDS TO EIGENVALUES

Let us consider an eigenvalue problem which is similar to the eigenvalue problem given by eqns (8) and (9)

$$\beta K(h) - \Omega^2 \gamma M(h) = 0 \tag{17}$$

where

$$\begin{aligned} h &= h(x) \quad a \le x \le b \\ \beta &= \beta(x) \quad a \le x \le b \\ \beta &= \gamma(x) \quad a \le x \le b \end{aligned} \tag{18}$$

with the same set of boundary conditions as the previous problem. The equivalent variational formulation of this problem is

$$\Omega_i^2 = \min \frac{\langle \beta h_i, Kh_i \rangle}{\langle \gamma h_i, Mh_i]} \quad i = 1, 2, \ldots, \infty \tag{19}$$

and the maximum–minimum formulation can be expressed as

$$\Omega_r^2 = \max \left\{ \min \frac{\langle \beta h_r, Kh_r \rangle}{\langle \gamma h_r, Mh_r \rangle} \right\} \quad r = 2, 3, \ldots, \infty \tag{20}$$

subject to r-1 constraints. The only difference of eqn (17) from eqn (8) is the weighting functions $\beta(x)$ and $\gamma(x)$. These two weighting functions approximate the weighting functions of the original problem, $u(x)$ and $\omega(x)$ respectively. The new weighting functions are selected so that eqn (17) is directly solvable and the exact values of the eigenvalues can be obtained. Also the following inequalities are assumed to hold true

$$\begin{aligned} &\langle 1, \beta \rangle < \langle 1, u \rangle \\ &\langle 1, \gamma \rangle > \langle 1, \omega \rangle. \end{aligned} \tag{21}$$

Now

$$\begin{aligned} \langle \beta h_i, Kh_i \rangle &= \langle uh_i, Kh_i \rangle + \langle (\beta - u)h_i, Kh_i \rangle \\ &\leq \langle uh_i, Kh_i \rangle + C_1 \langle 1, (\beta - u) \rangle \end{aligned} \tag{22}$$

where C_1 is the minimum of $\langle h_i, Kh_i \rangle$. C_1 is positive as K is a positive definite operator, and from eqn (21)

$$\langle 1, (\beta - u) \rangle < 0. \tag{23}$$

Therefore

$$\langle \beta h_i, Kh_i \rangle \leq \langle uh_i, Kh_i \rangle. \tag{24}$$

Also

$$\begin{aligned} \langle \gamma h_i, Mh_i \rangle &= \langle \omega h_i, Mh_i \rangle + \langle (\gamma - \omega)h_i, Mh_i \rangle \\ &\geq \langle \omega h_i, Mh_i \rangle + C_2 \langle 1, (\gamma - \omega) \rangle \end{aligned} \tag{25}$$

where C_2 is the minimum of $\langle h_i, Mh_i \rangle$. C_2 is positive as M is a positive definite operator, and from eqn (21)

$$\langle 1, (\gamma - \omega)] > 0. \tag{26}$$

Hence

$$\langle \gamma h_i, Mh_i \rangle \geq \langle \omega h_i, Mh_i \rangle. \tag{27}$$

For the first eigenvalue using eqns (10), (19), (24) and (27) we arrive at

$$\Omega_1^2 = \min \frac{\langle \beta h_1, Kh_1 \rangle}{\langle \gamma h_1, Mh_1 \rangle} \leq \frac{\langle \beta h_1, Kh_1 \rangle}{\langle \gamma h_1, Mh_1 \rangle} \leq \min \frac{\langle ug_1, Kg_1 \rangle}{\langle \omega g_1, Mg_1 \rangle} = \mu_1^2. \tag{28}$$

Hence

$$\Omega_1^2 \leq \mu_1^2. \tag{29}$$

For higher order eigenvalues

$$\Omega_i^2 = \max\left\{\min \frac{\langle \beta h_i, Kh_i\rangle}{\langle \gamma h_i, Mh_i\rangle}\right\} \leq \min \frac{\langle uh_i, Kh_i\rangle}{\langle \omega h_i, Mh_i\rangle}$$

$$\leq \max\left\{\min \frac{\langle ug_i, Kg_i\rangle}{\langle \omega g_i, Mg_i\rangle}\right\} = \mu_i^2 \quad i = 2, 3, \ldots, \infty. \tag{30}$$

Therefore

$$\Omega_i^2 \leq \mu_i^2 \quad i = 2, 3, \ldots, \infty. \tag{31}$$

Combining eqns (29) and (31)

$$\Omega_i^2 \leq \mu_i^2 \quad i = 1, 2, 3, \ldots, \infty. \tag{32}$$

Thus it has been shown that the eigenvalues of the second problem of which exact values are known constitute lower bounds to the eigenvalues of the original problem.

PROCEDURE

To assure the existence of an exact solution for the differential eigenvalue problem given by eqns (17) and (18) the weighting functions $\beta(x)$ and $\gamma(x)$ are assumed to be piecewise continuous step functions. That is, the region $a \leq x \leq b$ is divided into n subregions and both $\beta(x)$ and $\gamma(x)$ are set to be equal to a constant in each subregion.

$$\begin{aligned} &\gamma(x) = c_1^p, \ \beta(x) = d_1^p \ \text{in} \ e_1 \leq x < e_2 \\ &\gamma(x) = c_2^p, \ \beta(x) = d_2^p \ \text{in} \ e_2 \leq x < e_3 \\ &\quad \cdot \qquad \cdot \qquad \cdot \\ &\quad \cdot \qquad \cdot \qquad \cdot \\ &\quad \cdot \qquad \cdot \qquad \cdot \\ &\gamma(x) = c_n^p, \ \beta(x) = d_n^p \ \text{in} \ e_n \leq x \leq e_{n+1} \end{aligned} \tag{33}$$

where p is the order of the differential equation and e_i are the end points of the subregions. Therefore eqn (17) becomes a constant coefficient linear ordinary differential equation in each subregion, and it can be solved exactly. To satisfy eqn (21) c_i^p and d_i^p are chosen as

$$\begin{aligned} c_i^p &= \max(\omega) \ \text{in} \ e_i \leq x < e_{i+1} \\ d_i^p &= \min(u) \ \text{in} \ e_i \leq x < e_{i+1}. \end{aligned} \tag{34}$$

The subdivision points e_i are solved from the optimization

$$\min\left\{\langle 1, u\rangle - \sum_{i=1}^{n} \langle 1, d_i^p\rangle\right\}$$

or

$$\min\left\{\sum_{i=1}^{n} \langle 1, c_i^p\rangle - \langle 1, \omega\rangle\right\}. \tag{35}$$

After the e_i have been determined c_i^p and d_i^p are obtained from eqn (34), and eqn (17) is solved to yield a solution in each subregion. In this manner n different solutions are obtained. Then to obtain a continuous solution for the whole region $a \leq x \leq b$, the adjacent solutions h_{i-1} and h_i, and their derivatives up to $p - 1^{st}$ order are equated to each other at the subdivision points e_i, i.e.

$$h_{i-1}(e_i) = h_i(e_i)$$
$$h'_{i-1}(e_i) = h'_i(e_i)$$
$$\vdots$$
$$h_{i-1}^{p-1}(e_i) = h_i^{p-1}(e_i). \tag{36}$$

This set of equations coupled with the boundary conditions will give rise to pxn equations with $pxn + 1$ unknowns, one of the unknowns being the eigenvalue itself. To obtain a non-trivial solution the determinant of the coefficient matrix should be set equal to zero. This step will yield a transcendental equation in terms of Ω, and the roots of this equation will give lower bounds Ω_i^2.

NUMERICAL EXAMPLES

Graetz problem

The first example considered is the Graetz problem which is encountered in heat transfer, it is related to laminar flow in a round tube or flat conduit. The eigenvalue of this problem is proportional to heat transfer coefficient. This problem was solved as a check case as the exact eigenvalues are available in the literature[9]. The governing differential equation is

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dq}{d\rho}\right) + \mu^2(1-\rho^2)g = 0 \tag{37}$$

with the boundary conditions

$$g(1) = 0$$
$$\left.\frac{dg}{d\rho}\right|_{\rho=0} = 0. \tag{38}$$

Equation (37) can be written in operator form as

$$K(g) - \mu^2\omega M(g) = 0 \tag{39}$$

where

$$K = -\rho^2\frac{d^2}{d\rho^2} - \rho\frac{d}{d\rho}$$
$$M = \rho^2$$
$$\omega(\rho) = 1 - \rho^2. \tag{40}$$

For this problem Rayleigh's quotient can be written as

$$\upsilon^2 = \frac{\left\langle\rho\frac{d\phi}{d\rho}, \frac{d\phi}{d\rho}\right\rangle}{\langle\rho, (1-\rho^2)\phi^2\rangle}. \tag{41}$$

A trial function which satisfies the boundary condition is

$$\phi(\rho) = \sum_{i=1}^{n} A_i(1-\rho^{i+1}). \tag{42}$$

This trial function is substituted in eqn (41) and the Rayleigh–Ritz method[10] is applied to obtain upper bounds to eigenvalues. This method yields a matrix eigenvalue problem of which eigenvalues are upper bounds to the eigenvalues of the original problem. That matrix eigenvalue problem was solved for the first eigenvalue for n values ranging from 2 to 20. The results obtained are given in Table 1 and Fig. 1. This problem is a special case of the general eigenvalue problem defined by eqn (17), where the weighting function $u(x)$ is identically equal to 1.

To obtain lower bounds to this problem the region $0 \leq \rho \leq 1$ is subdivided into n subregions and eqn (39) is approximated by the following set of equations

$$K(g_i) - \Omega^2 \gamma M(g_i) = 0 \quad i = 1, 2, 3, \ldots n \tag{43}$$

Table 1. Graetz problem, first eigenvalue

Number of Terms	Lower Bound to the Eigenvalue	Upper Bound to the Eigenvalue	% Difference Between Upper and Lower Bound
2	6.386	7.317	13.588
3	6.567	7.315	10.777
4	6.690	7.313	8.898
5	6.779	7.313	7.579
6	6.846	7.313	6.597
7	6.898	7.313	5.841
8	6.939	7.313	5.248
9	6.973	7.313	4.760
10	7.002	7.313	4.345
11	7.026	7.313	4.003
12	7.046	7.313	3.719
13	7.064	7.313	3.464
14	7.079	7.313	3.252
15	7.093	7.313	3.054
16	7.105	7.313	2.885
17	7.116	7.313	2.731
18	7.126	7.313	2.590
19	7.135	7.313	2.464
20	7.143	7.313	2.352

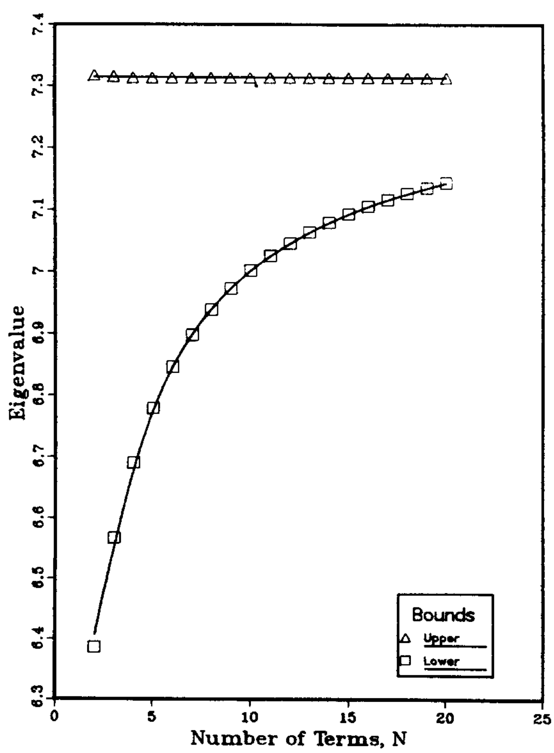

Fig. 1. Graetz problem, first eigenvalue.

where

$$\gamma(\rho) = c_i^2 \ \text{in} \ e_i \leq \rho < e_{i+1}, \ i = 1, 2, 3, \ldots, n. \tag{44}$$

Here the case for $n = 2$ will be explained in detail. For this case the subdivision points are e_1, e_2, e_3.

$$\begin{aligned} e_1 &= 0.0 \\ e_3 &= 1.0. \end{aligned} \tag{45}$$

The intermediate point e_2 is found from the condition

$$\min\left\{\sum_{i=1}^{2} \langle 1, c_i^2\rangle - \langle 1, \omega\rangle\right\} \tag{46}$$

where

$$c_i^2 = \max(1-\rho^2) \ \text{in} \ e_i \leq \rho < e_{i+1} \ i = 1, 2. \tag{47}$$

In the first subregion $e_1 \leq \rho < e_2$ eqn (43) becomes

$$K(g_1) - \Omega^2 c_1{}^2 M(g_1) = 0. \tag{48}$$

The second boundary condition given by eqn (38) is applicable to this region. The solution of eqn (48) is

$$g_1(\rho) = A_1 J_0(c_1\Omega\rho) + A_2 Y_0(c_1\Omega\rho). \tag{49}$$

Substituting in the boundary condition it is found that

$$A_2 = 0. \tag{50}$$

Therefore

$$g_1(\rho) = A_1 J_0(c_1\Omega\rho). \tag{51}$$

In the second subregion $e_2 \leq \rho \leq e_3$ eqn (43) becomes

$$K(g_2) - \Omega^2 c_2{}^3 M(g_2) = 0. \tag{52}$$

After substituting the first of boundary conditions (38) in, the solution to eqn (52) becomes

$$g_2(\rho) = A_4\left\{ y_0(c_2\Omega\rho) - \frac{Y_0(c_2\Omega)}{J_0(c_2\Omega)} J_0(c_2\Omega\rho)\right\}. \tag{53}$$

For this case the continuity conditions (36) become

$$\begin{aligned} g_1(e_2) &= g_2(e_2) \\ g_1'(e_2) &= g_2'(e_2). \end{aligned} \tag{54}$$

Equation (54) is a set of two simultaneous equations with three unknowns, one of the unknowns being Ω. To obtain a non-trivial solution the determinant of the coefficient matrix is set equal to zero. The transcendental equation obtained from this procedure is

$$\begin{aligned} &c_2 J_0(c_1\Omega e_2)\left\{\frac{Y_0(c_2\Omega)}{J_0(c_2\Omega)} J_1(c_2\Omega e_2) - Y_1(c_2\Omega e_2)\right\} \\ &\quad - c_1 J_1(c_1\Omega e_2)\left\{\frac{Y_0(c_2\Omega)}{J_0(c_2\Omega)} J_0(c_2\Omega e_2) - Y_0(c_2\Omega e_2)\right\} = 0. \end{aligned} \tag{55}$$

The smallest positive root of this equation will yield the lower bound for the first eigenvalue.

This problem was solved for $n = 2, 3, \ldots, 20$, and the results obtained are included in Table 1 and Fig. 1. The exact value of the first eigenvalue is given in Ref.[9]. The lower bound obtained here is close to, but better than the one given in Ref.[6], and much better than the one given in Ref.[11].

BUCKLING OF A VARIABLE THICKNESS COLUMN

The second example considered is the buckling of an exponentially varying thickness column. In this problem the eigenvalues are directly proportional to the critical buckling load. The governing equation of this problem is

$$EI\frac{d^2y}{dx^2} + Py = 0. \tag{56}$$

The thickness of the column is given as

$$s = s_0\, e^{-\alpha_1 x} \tag{57}$$

and it has a constant width t and length L. Therefore eqn (56) becomes

$$\frac{d^2y}{dx^2} + \frac{12P}{Ets_0^3}\, e^{3\alpha_1 x} = 0 \tag{58}$$

with the boundary conditions

$$\begin{aligned} y(0) &= 0 \\ y(L) &= 0. \end{aligned} \tag{59}$$

Making use of the variable transformation

$$\begin{aligned} \eta &= \frac{x}{L} \\ \alpha &= \alpha_1 L \end{aligned} \tag{60}$$

eqn (58) can be written as

$$\frac{d^2y}{d\eta^2} + \lambda^2 e^{3\alpha\eta} y = 0 \tag{61}$$

with

$$\begin{aligned} y(0) &= 0 \\ y(1) &= 0 \end{aligned} \tag{62}$$

where

$$\lambda^2 = \frac{12PL^2}{Ets_0^3}. \tag{63}$$

In operator form this problem can be expressed as

$$uK'(y) - \lambda^2 M'(y) = 0 \tag{64}$$

where

$$K' = -\frac{d^2}{d\eta^2},\ M' = 1,\ u = e^{-3\alpha\eta}. \tag{65}$$

Rayleigh's quotient for this case is

$$\upsilon^2 = -\frac{\left\langle e^{-3\alpha\eta}\phi, \frac{d^2\phi}{d\eta^2}\right\rangle}{\langle\phi, \phi\rangle}. \tag{66}$$

The trial function employed which satisfies the boundary conditions is

$$\phi(\eta) = \sum_{i=1}^{n} A_i \sin(i\pi\eta). \tag{67}$$

The upper bounds obtained by substituting the trial function into eqn (66) and solving it by Rayleigh–Ritz method[10] are given for selected α values in Tables 2 and 3, and Figs. 2 and 3.

To obtain lower bounds the form of eqn (61) is used. In operator form

$$K(y) - \lambda^2\omega M(y) = 0. \tag{68}$$

This equation is the same as the previous example. It is treated in the same manner as before. For $n = 2$ the transcendental equation obtained is

$$c_2 \sin(c_1\Omega e_2)\left\{\sin(c_2\Omega e_2) + \frac{\cos(c_2\Omega)}{\sin(c_2\Omega)}\cos(c_2\Omega e_2)\right\} - c_1\cos(c_1\Omega e_2)\left\{\frac{\cos(c_2\Omega}{\sin(c_2\Omega)}\sin(c_2\Omega e_2) - \cos(c_2\Omega e_2)\right\} = 0. \tag{69}$$

This problem, again, was solved for $n = 2, 3, \ldots, 20$. The results obtained for lower bounds for selected α values are included in Tables 2 and 3, and Figs. 2 and 3.

Table 2. Buckling problem, first eigenvalue $\alpha = 0.001$

Number of Terms	Lower Bound to the Eigenvalue	Upper Bound to the Eigenvalue	% Difference Between Upper and Lower Bound
2	9.849	9.854	0.051
4	9.851	9.854	0.030
6	9.852	9.854	0.020
8	9.853	9.854	0.010
10	9.853	9.854	0.010
12	9.853	9.854	0.010
14	9.853	9.854	0.010
16	9.853	9.854	0.010
18	9.854	9.854	0.000
20	9.854	9.854	0.000

Table 3. Buckling problem, first eigenvalue $\alpha = 0.100$

Number of Terms	Lower Bound to the Eigenvalue	Upper Bound to the Eigenvalue	% Difference Between Upper and Lower Bound
2	8.064	8.474	4.958
4	8.225	8.474	2.982
6	8.295	8.474	2.135
8	8.334	8.474	1.666
10	8.359	8.474	1.366
12	8.377	8.474	1.151
14	8.390	8.474	0.996
16	8.399	8.474	0.889
18	8.407	8.474	0.794
20	8.413	8.474	0.722

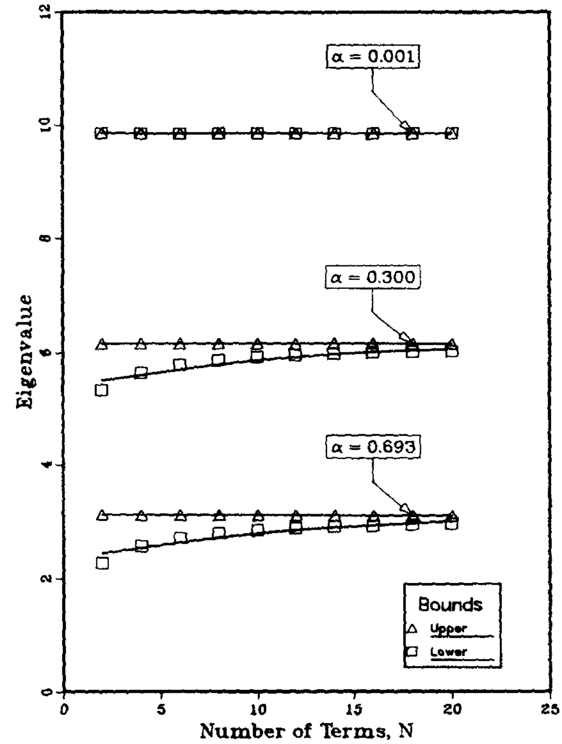

Fig. 2. Buckling problem, first eigenvalue.

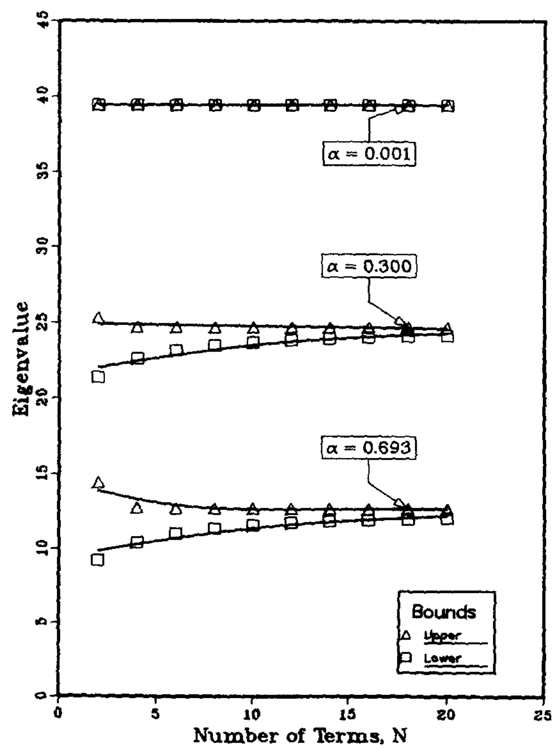

Fig. 3. Buckling problem, second eigenvalue.

TRANSVERSE VIBRATION OF A VARIABLE THICKNESS BEAM

The third example considered is the bending vibration of a simply supported beam with exponentially varying thickness. The eigenvalues are proportional to the natural frequencies of the beam. The governing equation is

$$\frac{\partial^2}{\partial x^2}\left\{EI\,\frac{\partial^2 y}{\partial x^2}\right\} = -\rho A\,\frac{\partial^2 y}{\partial \tau^2} \tag{70}$$

with the boundary conditions

$$\begin{aligned} &y(0,\tau)=0 \qquad y(L,\tau)=0\\ &\left.\frac{\partial^2 y}{\partial x^2}\right|_{x=0}=0 \quad \left.\frac{\partial^2 y}{\partial x^2}\right|_{x=L}=0. \end{aligned} \tag{71}$$

Here it is assumed that the beam has a length L and width t. Its thickness is given by

$$s = s_0\,e^{-\alpha_1 x}. \tag{72}$$

By applying separation of variables of the form

$$y(x,\tau) = \Gamma(x)F(\tau) \tag{73}$$

eqn (70) becomes

$$\frac{d^2}{dx^2}\left\{e^{-3\alpha_1 x}\,\frac{d^2\Gamma}{dx^2}\right\} = \frac{12\rho}{Es_0^2}\,e^{-\alpha_1 x}\,\lambda_*^4\Gamma. \tag{74}$$

Introducing the variable transformation

$$\eta = \frac{x}{L};\ \alpha = \alpha_1 L \tag{75}$$

eqn (74) becomes

$$\frac{d^2}{d\eta^2}\left\{e^{-3\alpha\eta}\,\frac{d^2\Gamma}{d\eta^2}\right\} = \lambda^4\,e^{-\alpha\eta}\,\Gamma \tag{76}$$

with the boundary conditions

$$\begin{aligned} &\Gamma(0)=0 \qquad \Gamma(1)=0\\ &\left.\frac{d^2\Gamma}{d\eta^2}\right|_{\eta=0}=0 \quad \left.\frac{d^2\Gamma}{d\eta^2}\right|_{\eta=1}=0 \end{aligned} \tag{77}$$

where

$$\lambda^4 = \frac{12\rho L^4}{Es_0^2}\,\lambda_*^4. \tag{78}$$

In operator form eqn (76) becomes

$$K(\Gamma) - \lambda^4\omega M(\Gamma) = 0 \tag{79}$$

where

$$\begin{aligned} K &= \frac{d^2}{d\eta^2}\left\{u\,\frac{d^2}{d\eta^2}\right\}\\ M &= 1\\ u &= e^{-3\alpha\eta}\\ \omega &= e^{-\alpha\eta}. \end{aligned} \tag{80}$$

Rayleigh's quotient for this problem can be written as

$$\nu^4 = \frac{\left\langle e^{-3\alpha\eta}\frac{d^2\phi}{d\eta^2}, \frac{d^2\phi}{d\eta^2}\right\rangle}{\langle e^{-\alpha\eta}\phi, \phi\rangle}. \tag{81}$$

An admissible function is

$$\phi(\eta) = \sum_{i=1}^{n} A_i \sin(i\pi\eta). \tag{82}$$

It is used as a trial function in the Rayleigh–Ritz method[10] to obtain upper bounds to the exact eigenvalues. Upper bounds obtained this way are tabulated and plotted, for selected α values, in Tables 4 and 5, and Figs. 4 and 5.

Applying the lower bounds method the region $0 \leq \eta \leq 1$ is divided into n subregions and the governing equation is approximated as

$$\beta K^*(\Gamma_i) - \Omega^4\gamma M(\Gamma_i) = 0 \quad i = 1, 2, 3, \ldots, n \tag{83}$$

Table 4. Transverse bending vibrations, first eigenvalue $\alpha = 0.001$

Number of Terms	Lower Bound to the Eigenvalue	Upper Bound to the Eigenvalue	% Difference Between Upper and Lower Bound
2	88.032	97.311	10.013
3	91.993	97.311	5.618
4	93.843	97.311	3.628
5	94.829	97.311	2.584
6	95.416	97.311	1.967
7	95.976	97.311	1.381
8	96.058	97.312	1.297
9	96.248	97.318	1.106
10	96.391	97.311	0.950
11	96.502	97.311	0.835
12	96.591	97.311	0.743
13	96.662	97.311	0.669
14	96.722	97.311	0.607
15	96.771	97.311	0.556
16	96.814	97.311	0.512
17	96.850	97.311	0.475
18	96.882	97.311	0.442
19	96.909	97.311	0.414
20	96.934	97.311	0.388

Table 5. Transverse bending vibrations, first eigenvalue $\alpha = 0.100$

Number of Terms	Lower Bound to the Eigenvalue	Upper Bound to the Eigenvalue	% Difference Between Upper and Lower Bound
2	76.636	88.252	14.090
3	80.973	88.252	8.603
4	83.086	88.252	6.030
5	84.267	88.252	4.620
6	85.004	88.252	3.749
7	85.504	88.252	3.163
8	85.864	88.251	2.742
9	86.135	88.253	2.429
10	86.347	88.251	2.181
11	86.517	88.248	1.981
12	86.650	88.244	1.823
13	86.772	88.259	1.699
14	86.870	88.230	1.553
15	86.954	88.238	1.466
16	87.028	88.212	1.351
17	87.092	88.263	1.336
18	87.149	88.164	1.158
19	87.199	88.196	1.137
20	87.244	88.262	1.160

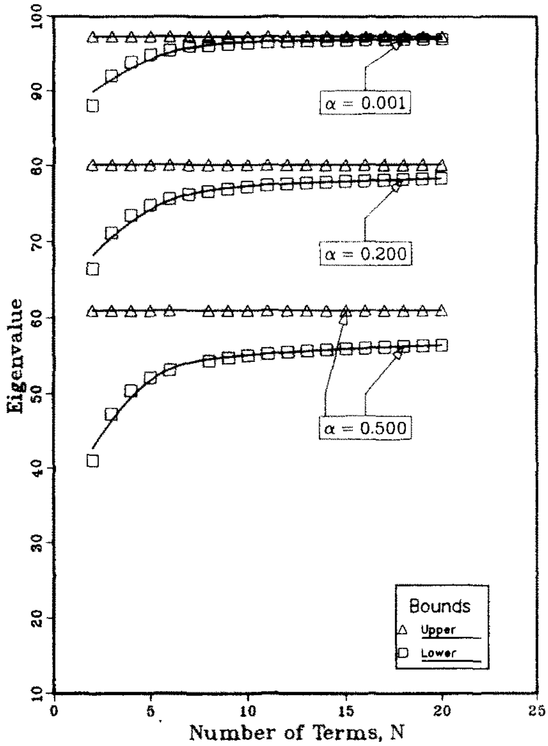

Fig. 4. Transverse bending vibration problem, first eigenvalue.

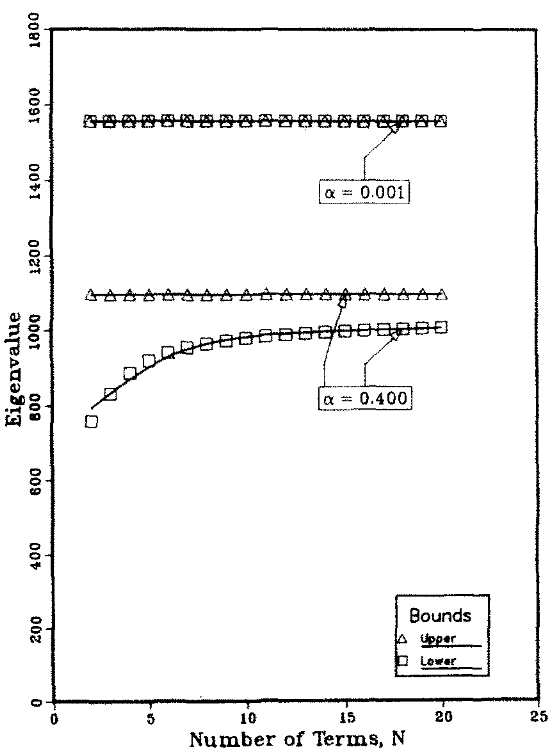

Fig. 5. Transverse bending vibration problem, second eigenvalue.

where

$$K^* = \frac{d^4}{d\eta^4} \quad M = 1$$
$$\gamma = c_i^4 \text{ in } e_i \leq \eta < e_{i+1}$$
$$\beta = d_i^4 \text{ in } e_i \leq \eta < e_{i+1}. \tag{84}$$

For $n = 2$ the intermediate subdivision point is solved from

$$\min\left\{\langle 1, u\rangle - \sum_{i=1}^{2} \langle 1, d_i^4\rangle\right\} \tag{85}$$

where

$$\begin{matrix} c_i^4 = \max(e^{-\alpha\eta}) \\ d_i^4 = \min(e^{-3\alpha\eta}) \end{matrix} \quad e_i \leq \eta < e_{i+1} \; i = 1, 2. \tag{86}$$

Substituting in eqn (83)

$$\frac{d^4\Gamma}{d\eta^4} - k_i^4\Omega^4\Gamma = 0 \quad i = 1, 2 \tag{87}$$

where

$$k_i^4 = \frac{d_i^4}{c_i^4} \quad i = 1, 2. \tag{88}$$

The solution to eqn (87) in the first subregion $e_1 \leq \eta < e_2$ is

$$\Gamma_1(\eta) = A_1 \sin(k_1\Omega\eta) + A_2 \cos(k_1\Omega\eta) + A\,3 \sinh(k_1\Omega\eta) + A_4 \cosh(k_1\Omega\eta). \tag{89}$$

Making use of proper boundary conditions eqn (89) becomes

$$\Gamma_1(\eta) = A_1 \sin(k_1\Omega\eta) + A_3 \sinh(k_1\Omega\eta). \tag{90}$$

In the second subregion $e_2 \leq \eta \leq e_3$, after substituting in the proper boundary conditions the solution becomes

$$\Gamma_2(\eta) = A_5\left\{\sin(k_2\Omega\eta) - \frac{\sin(k_2\Omega)}{\cos(k_2\Omega)}\cos(k_2\Omega\eta)\right\}$$
$$+ A_7\left\{\sinh(k_2\Omega\eta) - \frac{\sinh(k_2\Omega)}{\cosh(k_2\Omega}\cosh(k_2\Omega\eta)\right\}. \tag{91}$$

The continuity conditions given by eqn (36) become

$$\Gamma_1(e_2) = \Gamma_2(e_2)$$
$$\Gamma_1'(e_2) = \Gamma_2'(e_2)$$
$$\Gamma_1''(e_2) = \Gamma_2''(e_2)$$
$$\Gamma_1'''(e_2) = \Gamma_2'''(e_2). \tag{92}$$

Equation (92) yields a set of simultaneous nonlinear equations. As previously by setting the determinant of the coefficient matrix to zero a transcendental equation in terms of Ω is obtained. Then that equation is solved for Ω^4 which are the lower bounds to the eigenvalues. The value of n changed from 2 to 20, and the results are tabulated and plotted, for selected α values, in Tables 4 and 5 and Figs. 4 and 5.

CONCLUSION

The results obtained for the Graetz problem when compared with the exact values[9] indicate that both upper and lower bounds methods yield very satisfactory results, even when n is small.

For the buckling and transverse beam vibration eigenvalue problems the results indicate that the gap between the upper and lower bounds are quite small for most values of the parameter α [12]. As α increases, more subdivisions are necessary in order to decrease the gap between the upper and lower bounds. In all cases however, the gap between the upper and lower bounds is small enough so that the buckling loads and natural frequencies of vibration can be estimated with a high degree of accuracy for most practical engineering applications. This is generally not the case when other numerical solution methods are used since only an approximate result is obtained and no error estimate is provided.

The main goal of developing a lower bounds method which can be easily applied and which will be the counterpart of the Rayleigh–Ritz method is achieved. The lower bounds display an asymptotic approach to the true value as the number of subdivisions is increased.

The differential equations considered here are the type of equations that are most frequently encountered in practical engineering problems. This fact assures that the lower bounds method developed is applicable to a large number of different problems.

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